INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2008; **56**:1351–1357
Published online 19 September 2007 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/fld.1587

An essentially non-oscillatory Crank–Nicolson procedure for incompressible Navier–Stokes equations

Chung-Ki Cho[1] and Seongjai Kim[2,*,†]

[1]*Department of Mathematics, Soonchunhyang University, Asan, Choongnam 336-745, South Korea*
[2]*Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762-5921, U.S.A.*

SUMMARY

This article introduces an *essentially non-oscillatory Crank–Nicolson* (ENO-CN) scheme for the numerical solution of unsteady incompressible Navier–Stokes (NS) equations. ENO-CN utilizes the second-order upwind ENO scheme for the approximation of the convection term in the explicit step, while everything else is discretized by the central finite volume method. It has been numerically verified for NS equations in 2D that the resulting algorithm is non-oscillatory for a large range of Reynolds numbers and less dissipative than the conventional upwind methods and deferred correction methods. The new approach can be extended for 3D problems straightforwardly. Copyright © 2007 John Wiley & Sons, Ltd.

Received 22 March 2007; Revised 5 July 2007; Accepted 6 July 2007

KEY WORDS: incompressible Navier–Stokes equations; essentially non-oscillatory Crank–Nicolson (ENO-CN) scheme; deferred correction; projection method

1. INTRODUCTION

Let Ω be a bounded domain in the two-dimensional (2D) space and $J=(0,T]$, $T>0$. Consider the unsteady, incompressible, viscous Navier–Stokes (NS) equations:

$$\begin{aligned} &\text{(a)}\ \frac{\partial(\rho v_j)}{\partial x_j}=0, \quad (\mathbf{x},t)\in\Omega\times J \\ &\text{(b)}\ \frac{\partial(\rho v_i)}{\partial t}+\frac{\partial(\rho v_j v_i)}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ij}}{\partial x_j}+\rho g_i, \quad (\mathbf{x},t)\in\Omega\times J \end{aligned} \tag{1}$$

*Correspondence to: Seongjai Kim, Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762-5921, U.S.A.
†E-mail: skim@math.msstate.edu

Contract/grant sponsor: Soonchunhyang University
Contract/grant sponsor: NSF; contract/grant number: DMS-0630798, DMS-0609815

where ρ is the density, $\mathbf{v}=(v_1, v_2)$ denotes the velocity, p is the static pressure, $\mathbf{g}=(g_1, g_2)$ is the gravity vector, and $\boldsymbol{\tau}=(\tau_{ij})$ is the viscous part of the stress tensor defined by

$$\tau_{ij}=\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)+\left(\kappa-\frac{2}{3}\mu\right)\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\right)\delta_{ij}$$

Here, μ and κ are, respectively, the shear coefficient of viscosity and the bulk coefficient of viscosity, and δ_{ij} denotes the Kronecker delta. We have adopted the Einstein summation convention.

It is now well known that for large Reynolds numbers ($Re:=\rho/\mu$), the standard central schemes for the convection term easily introduce non-physical oscillation, while upwind schemes smear out the sharp-fronts of the solution. Thus, in the simulation of such convection-dominated flows, the most challenging issue is to suppress both non-physical oscillation and numerical dissipation.

Various numerical methods have been suggested to overcome these difficulties. Among others, interesting methods are the streamline diffusion methods [1] and the discontinuous Galerkin methods [2, 3]. Methods evolving the numerical solution along characteristics are semi-Lagrangian methods [4, references therein], ELLAM [5, 6], and particle-mesh methods [7]. For solving hyperbolic conservation laws and Hamilton–Jacobi equations, high-resolution methods have been studied; see MUSCL [8, 9], QUICK [10], and ENO [11, 12], which are often integrated with explicit time-stepping procedures to be total variation diminishing (TVD) schemes. However, the approximate solutions of these relatively new methods may still exhibit either over- or under-shoots for convection-dominant flow problems.

This article is concerned with an *essentially non-oscillatory Crank–Nicolson* (ENO-CN) scheme for the numerical solution of (1). The standard Crank–Nicolson (CN) method is strictly non-dissipative but *easily* oscillatory for non-smooth solutions of convection-dominated flows. To suppress the non-physical oscillation, we will replace the central scheme applied to the explicit convection term by one of the high-order upwind schemes. We have found that the second-order essentially non-oscillatory scheme (ENO2) [11, 12] is particularly appropriate for high-resolution numerical solutions for convective flows.

The organization of this article is as follows. The next section briefly reviews the second-order projection method of the NS equation (1) suggested by Choi–Moin [13]. In the same section, we introduce an ENO-CN procedure which is more effective than conventional deferred correction methods. Section 3 shows numerical examples which simulate viscous flows in lid-driven cavities. Our development and experiments are concluded in Section 4.

2. ALGORITHMS

Partition J into $0=t^0<t^1<\cdots<t^{N_t}=T$, for a positive integer N_t. Define $\Delta t^n=t^n-t^{n-1}$, and $g^n(\mathbf{x})=g(\mathbf{x}, t^n)$ for a function g of independent variables $(\mathbf{x}, t)\in\Omega\times J$. For a simple presentation, we define an operator $H_i(\mathbf{v})$ representing discretized convective and diffusive terms:

$$H_i(\mathbf{v})=\frac{\delta(\rho v_j v_i)}{\delta x_j}-\frac{\delta\tau_{ij}}{\delta x_j}$$

DOI: 10.1002/fld

Hereafter spatial approximations would be formally expressed; we will adopt second-order finite volume methods. Then, the CN scheme for (1b) can be formulated as follows:

$$\frac{(\rho v_i)^n - (\rho v_i)^{n-1}}{\Delta t^n} + \frac{1}{2}[H_i(\mathbf{v}^n) + H_i(\mathbf{v}^{n-1})] = -\frac{1}{2}\left(\frac{\delta p^n}{\delta x_i} + \frac{\delta p^{n-1}}{\delta x_i}\right) + \rho g_i \tag{2}$$

When the spatial derivatives are approximated by second-order schemes, scheme (2) shows the truncation error of $\mathcal{O}((\Delta t^n)^2 + |\Delta \mathbf{x}|^2)$ for smooth solutions.

One approach for the simulation of NS flows is to adopt one of the projection methods [14, 15]. Here, we briefly review the second-order projection method for (2) suggested by Choi–Moin [13], which consists of three fractional steps for the advancement of a time level:

$$\begin{aligned}
&\text{(a)}\ \frac{(\rho v_i)^* - (\rho v_i)^{n-1}}{\Delta t^n} + \frac{1}{2}[H_i(\mathbf{v}^*) + H_i(\mathbf{v}^{n-1})] = -\frac{\delta p^{n-1}}{\delta x_i} + \rho g_i \\
&\text{(b)}\ \frac{(\rho v_i)^{**} - (\rho v_i)^*}{\Delta t^n} = \frac{1}{2}\frac{\delta p^{n-1}}{\delta x_i} \\
&\text{(c)}\ \frac{(\rho v_i)^n - (\rho v_i)^{**}}{\Delta t^n} = -\frac{1}{2}\frac{\delta p^n}{\delta x_i}
\end{aligned} \tag{3}$$

In the first step of the above algorithm, the velocity $\mathbf{v}^*$ is advanced with the pressure explicitly treated. Note that this step is nonlinear and, therefore, it should be solved iteratively up to the satisfaction of a narrow tolerance. In the second step, half the old pressure gradient is removed; the final step computes the velocity in the new time level, after solving the Poisson equation,

$$\frac{\delta}{\delta x_i}\left(\frac{\delta p^n}{\delta x_i}\right) = \frac{2}{\Delta t^n}\frac{\delta(\rho v_i)^{**}}{\delta x_i} \tag{4}$$

which is obtained to fill the requirement that the new velocity should satisfy the continuity equation (1a).

The nonlinear problem (3a) can be linearized by, e.g. the Newton method and the Picard iteration; we will employ the Picard iteration. Special care must be taken in order for the convection term not to introduce non-physical oscillation nor excessive numerical dissipation. One method is the *deferred correction* [16]: Given $\{v_i^{*,0}\}$, find $\{v_i^{*,k}\}$, $k \geqslant 1$, by solving

$$\begin{aligned}
&\frac{(\rho v_i)^{*,k} - (\rho v_i)^{n-1}}{\Delta t^n} + \frac{1}{2}\nabla_h^{\mathrm{UD}} \cdot (\rho v_i^{*,k}\mathbf{v}^{*,k-1}) - \frac{1}{2}\nabla_h^{\mathrm{CD}} \cdot \boldsymbol{\tau}^{*,k} \\
&\quad = -\frac{1}{2}[(1-\beta)\nabla_h^{\mathrm{UD}} \cdot (\rho v_i^{n-1}\mathbf{v}^{n-1}) + \beta\nabla_h^{\mathrm{CD}} \cdot (\rho v_i^{n-1}\mathbf{v}^{n-1})] \\
&\qquad + \frac{1}{2}\nabla_h^{\mathrm{CD}} \cdot \boldsymbol{\tau}^{n-1} + F_i + \beta\frac{1}{2}[\nabla_h^{\mathrm{UD}} \cdot (\rho v_i^{*,k-1}\mathbf{v}^{*,k-1}) - \nabla_h^{\mathrm{CD}} \cdot (\rho v_i^{*,k-1}\mathbf{v}^{*,k-1})]
\end{aligned} \tag{5}$$

where ∇_h^{UD} and ∇_h^{CD} are, respectively, the upwind (first-order) and central (second-order) approximation operators, F_i denotes the right side of (3a), and β is a *blending* parameter ($0 \leqslant \beta \leqslant 1$). The solution in the previous time level can be utilized as the initial value, i.e. $v_i^{*,0} = v_i^{n-1}$.

DOI: 10.1002/fld

Higher resolution can be obtained for a larger β, but the blending parameter is not allowed to be near one. It has been observed from various numerical experiments that blending parameters between 0.4 and 0.75 result in a non-oscillatory solution having a reasonably high resolution. However, the choice of β is often problematic; see [16] for details.

Now, we are ready to introduce a new method called the ENO-CN procedure for (3a):

$$\frac{(\rho v_i)^{*,k} - (\rho v_i)^{n-1}}{\Delta t^n} + \frac{1}{2}\nabla_h^{\mathrm{CD}} \cdot (\rho v_i^{*,k} \mathbf{v}^{*,k-1}) - \frac{1}{2}\nabla_h^{\mathrm{CD}} \cdot \boldsymbol{\tau}^{*,k}$$
$$= -\frac{1}{2}\nabla_h^{\mathrm{ENO}} \cdot (\rho v_i^{n-1} \mathbf{v}^{n-1}) + \frac{1}{2}\nabla_h^{\mathrm{CD}} \cdot \boldsymbol{\tau}^{n-1} + F_i \tag{6}$$

where ∇_h^{ENO} is one of high-order upwind ENO schemes [12]. Here, we choose the second-order ENO scheme (ENO2).

Note that ENO-CN (6) differs from (5) in the way of blending. In ENO-CN, the convection term has been approximated by a high-order upwind ENO scheme only in the *explicit* step. It should be noticed that ENO-CN introduces no extra parameter.

In order to show the effectiveness of ENO-CN, we apply the new algorithm to the following one-way wave equation (of no diffusion):

$$u_t + u_x = 0, \quad (x, t) \in (-2, 14) \times (0, 10]$$
$$u(x, 0) = \begin{cases} 1 & \text{if } -1 \leqslant x \leqslant 0 \\ x - 1 & \text{if } 1 \leqslant x < 2 \\ 3 - x & \text{if } 2 \leqslant x < 3 \\ 0 & \text{elsewhere} \end{cases} \tag{7}$$

Figure 1 depicts one-way wave solutions at $t = 10$ computed by three different methods: ENO2 with the second-order Runge–Kutta scheme (ENO2-RK2), ENO3 with a TVD third-order Runge–Kutta scheme (ENO3-RK3), and the new algorithm (ENO-CN or ENO2-CN). Set $h_x = 0.01$

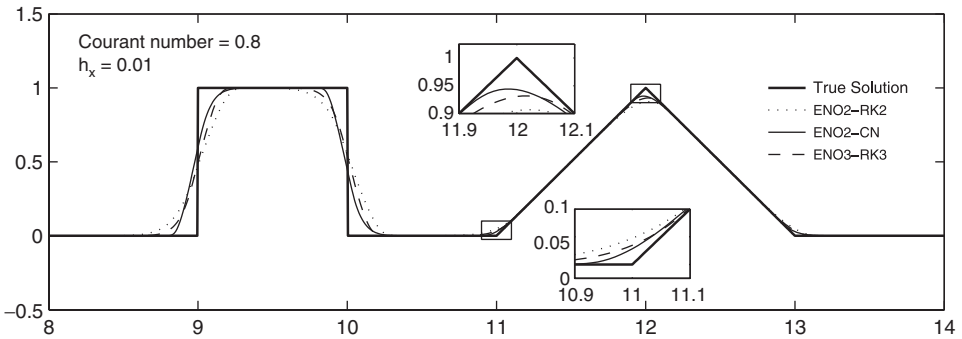

Figure 1. One-way wave solutions at $t = 10$, shown on the interval [8, 14].

DOI: 10.1002/fld

and the Courant number, $\Delta t/h_x = 0.8$. As one can see from the figure, ENO-CN has introduced no non-physical oscillation and simulated the solution better than ENO2-RK2 and ENO3-RK3. Note that ENO-CN has a second-order accuracy at best; in practice, it is better than ENO3-RK3, a third-order method.

The above example shows an interesting aspect of the CN procedure. It can effectively suppress dissipation as far as the numerical solution does not involve non-physical oscillation. (As mentioned before, the CN method is non-dissipative for all frequency components of smooth solutions.) For the numerical solution of linear convection-dominated flow problems in 2D, ENO-CN has been analyzed to be stable when the Courant number is set not larger than one, and it in practice has proved an accuracy order of 1.6–1.9 in the least-squares (L^2) norm [17].

When ENO-CN is applied for the numerical solution of NS equations, we do not know its *mathematical* properties in accuracy and stability. However, it has been verified from various numerical examples that it is more accurate than the deferred correction method (5) and stable for all choices of timestep size with which the Picard iteration converges. We will show an example in the following section.

3. NUMERICAL EXPERIMENTS

Let $\Omega = (0, 1)^2$, the unit square, and $\Gamma = \partial\Omega$, its boundary. Consider two lid-driven cavity problems of which the boundary conditions are specified as follows: for $t \in J = (0, T]$, $T > 0$,

$$\text{CP1}: \mathbf{v}(x, y, t) = \begin{cases} (1, 0) & \text{on } \{(x, y) \in \Gamma | y = 1\} \\ (0, 0) & \text{else on } \Gamma \end{cases}$$

$$\text{CP2}: \mathbf{v}(x, y, t) = \begin{cases} (1, 0) & \text{on } \{(x, y) \in \Gamma | y = 1\} \\ (\min(1, t/10), 0) & \text{on } \{(x, y) \in \Gamma | y = 0\} \\ (0, 0) & \text{else on } \Gamma \end{cases} \tag{8}$$

Set $T = 20$. The density $\rho \equiv 1$ and the shear coefficient of viscosity $\mu = 3.125 \times 10^{-4}$. (So the Reynolds number $Re = 3200$.) The bulk coefficient of viscosity is set as $\kappa = \frac{2}{3}\mu$, for simplicity. The initial values are all set to zero: $\mathbf{v}(x, y, 0) = p(x, y, 0) = 0$, $(x, y) \in \Omega$. The domain is partitioned into uniform $N_x \times N_y$ cells, and the timestep size Δt is set as $\frac{2}{3}\Delta x$. Define vertical cross sections as

$$X_\alpha = \{(x, y) \in \Omega | x = \alpha\}, \quad 0 \leqslant \alpha \leqslant 1$$

Figure 2 shows solution profiles of the horizontal velocity component (v_1), for $N_x = N_y = 200$ and $N_x = N_y = 400$, on $X_{0.5}$ for CP1 (left) and on $X_{0.8}$ for CP2 (right). For the deferred correction, we set two values for β: 0.0 and 0.5. It is apparent from both the figures that the solutions converge to the one obtained by ENO-CN with $N_x = N_y = 400$, as the mesh is refined. ENO-CN has resulted in more accurate and sharper velocity profiles than the deferred correction; its solution on the coarse mesh ($N_x = N_y = 200$ and $\Delta t = \frac{1}{250}$) reveals a better accuracy than that of the deferred correction method on the fine mesh ($N_x = N_y = 400$ and $\Delta t = \frac{1}{500}$). Note that ENO-CN has already converged on the coarse mesh.

DOI: 10.1002/fld

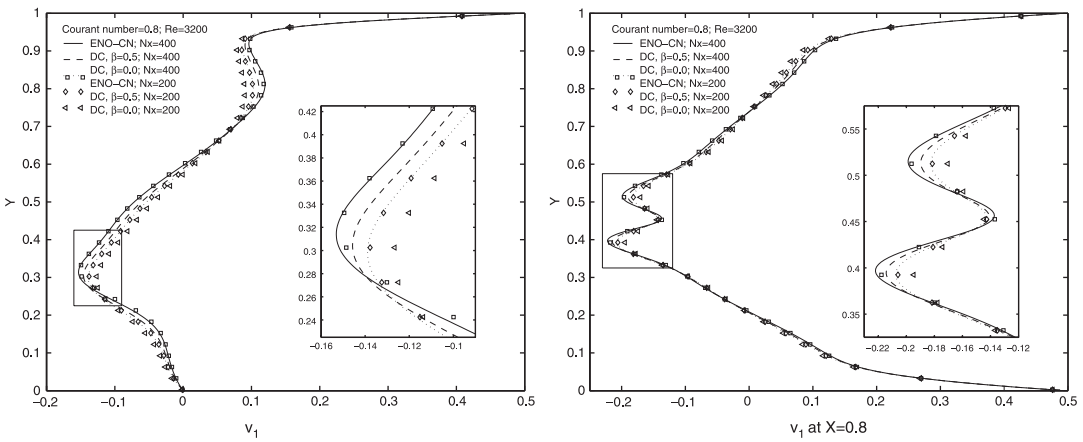

Figure 2. Profiles of the horizontal velocity component (v_1) on $X_{0.5}$ for CP1 (left) and on $X_{0.8}$ for CP2 (right).

It has been verified from various examples that ENO-CN is more accurate and resolves sharper velocity profiles than the conventional deferred correction methods. ENO-CN has introduced no non-physical oscillation for timestep sizes of the Courant number not larger than one.

4. CONCLUSIONS

We have introduced an *essentially non-oscillatory Crank–Nicolson* (ENO-CN) procedure for the numerical solution of unsteady incompressible Navier–Stokes equations. In order to effectively minimize numerical dissipation and suppress non-physical oscillation arising in numerical solutions of large Reynolds numbers, the new procedure incorporates the second-order upwind ENO scheme for the approximation of the convection term in the explicit step, while everything else is discretized by the central finite volume method. Our resulting algorithm has proved superior to the conventional upwind methods and deferred correction methods.

ACKNOWLEDGEMENTS

The authors are grateful for constructive and helpful suggestions from two anonymous referees. The work of the first author is supported by Soonchunhyang University Research Year Program; that of the second author is supported in part by NSF grants DMS-0630798 and DMS-0609815.

REFERENCES

1. Hughes TJ, Brooks A. A multidimensional upwind scheme with no crosswind diffusion. In *Finite Element Methods for Convection Dominated Flows*, vol. 34, Hughes TJ (ed.). AMD, ASME: NY, 1979.
2. Hu C, Shu CW. A discontinuous Galerkin finite element method for Hamilton–Jacobi equations. *SIAM Journal on Scientific Computing* 1999; **21**:666–690.
3. Johnson C, Pitkaranta J. An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation. *Mathematics of Computation* 1986; **46**:1–26.

DOI: 10.1002/fld

4. Staniforth A, Côté J. Semi-Lagrangian integration schemes for atmospheric models—a review. *Monthly Weather Review* 1991; **119**:2206–2223.
5. Celia M, Russell T, Herrera I, Ewing R. An Euclidean–Lagrangian localized adjoint method for the advection-diffusion equation. *Advances in Water Resources* 1990; **13**:197–206.
6. Wang H, Ewing R, Qin G, Lyons S, Al-Lawatia M, Man S. A family of Euclidean–Lagrangian localized adjoint methods for multi-dimensional advection–reaction equations. *Journal of Computational Physics* 1999; **152**:120–163.
7. Coppa G, Lapenta G, Dellanpiana G, Donato F, Riccardo V. Blob method for kinetic plasma simulation with variable-size particles. *Journal of Computational Physics* 1996; **127**:268–284.
8. van Leer B. Towards the ultimate conservative difference scheme. II. Monotonicity and conservation combined in a second-order scheme. *Journal of Computational Physics* 1974; **14**:361–370.
9. van Leer B. Towards the ultimate conservative difference scheme. V. A second-order sequel to Godunov's method. *Journal of Computational Physics* 1979; **32**:101–136.
10. Leonard B. A stable and accurate convective modelling procedure based on quadratic upstream interpolation. *Computer Methods in Applied Mechanics and Engineering* 1979; **19**:59–98.
11. Osher S, Sethian J. Fronts propagating with curvature dependent speed: algorithms based on Hamilton–Jacobi formulations. *Journal of Computational Physics* 1988; **79**:12–49.
12. Osher S, Shu CW. High-order essentially nonoscillatory schemes for Hamilton–Jacobi equations. *SIAM Journal on Numerical Analysis* 1991; **28**:907–922.
13. Choi H, Moin P. Effects of the computational time step on numerical solutions of turbulent flow. *Journal of Computational Physics* 1994; **113**:1–4.
14. Chorin A. The numerical solution of the Navier–Stokes equations for an incompressible fluid. *Bulletin of the American Mathematical Society* 1967; **73**:928.
15. Temam R. Sur l'approximation de la solution des equations de Navier–Stokes par la méthode des pas fractionaires II. *Archive for Rational Mechanics and Analysis* 1969; **33**:377–385.
16. Ferziger J, Peric M. *Computational Methods for Fluid Dynamics* (2nd edn). Springer: Berlin, Heidelberg, New York, 1999.
17. Alwehebi A, Williams N, Kim S. An essentially non-oscillatory Crank–Nicolson procedure for the simulation of convection-dominated flows, submitted.

DOI: 10.1002/fld